Solving and Simulating Nonlinear Stochastic Equilibrium Models in Frequency Domain

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Abstract

I propose a method of endogenous calibration of nonlinear stochastic equilibrium models, by taking advantage of spectral properties of economic data. Model parameters are determined by minimizing a distance metric specified in frequency domain. Several weight functions are discussed and their usefulness is assessed. Both univariate and multivariate settings are considered. Models' statistical testing strategies are discussed.

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1. Introduction

Non Linear-Quadratic (LQ) models are often used in studying optimal decision-making under uncertainty, but they are hard to solve because the method of stochastic calculus of variation (Kushner, 1965a, 1965b) leads to *nonlinear* stochastic difference equations. A widely used solution method (Kydland and Prescott, 1982) replaces the non-LQ model with its LQ approximation. The model is then calibrated and used to generate time series, which are compared to actual data. To calibrate, most studies “import” some parameters from other studies, while others are found by matching second moments of actual and simulated data.

Parameter importation from other studies, however, can bias the results because their econometric identification relies on different population moment conditions. I, therefore, determine these parameters endogenously, by minimizing a distance metric specified in frequency domain. Macroeconomic time series exhibit a remarkably uniform and typical frequency domain behavior (Granger, 1966; Levy and Dezhbakhsh, 2003a, 2003b) and the proposed method takes advantage of it. The existing studies are limited to cyclical frequencies. I can focus on any frequency, which makes this method more general, and potentially more useful.

2. The Method

Consider a typical rational expectations stochastic equilibrium model, where the goal is to

$$(1) \qquad \max E_0 \sum_{t=1}^{\infty} R^t U(\mathbf{Y}_t, \mathbf{\Theta}) \quad \text{s.t. } \mathbf{G}(\mathbf{Y}_t, \mathbf{Y}_{t-1}, ..., \mathbf{X}_t, \mathbf{X}_{t-1}, ..., \boldsymbol{\eta}_t, \mathbf{\Psi}) = \mathbf{0},$$

where U is the utility function, $\mathbf{G}$ is a vector of constraints, $\mathbf{Y}$ is a vector of choice variables, $\mathbf{X}$ is a vector of exogenous variables, $\boldsymbol{\eta}$ is a vector of stochastic shocks, $\mathbf{\Theta}$ and $\mathbf{\Psi}$ are vectors of utility and production function parameters, and R is the discount factor.

Suppose that the model is solved by some method and its solution is given by

$$(2) \qquad \mathbf{Y}_t = \mathbf{f}(\mathbf{Y}_{t-1}, \mathbf{Y}_{t-2}, ..., \mathbf{X}_t, \mathbf{X}_{t-1}, ..., \boldsymbol{\eta}_t, \boldsymbol{\eta}_{t-1}, ..., \mathbf{\Phi}),$$

where $\mathbf{\Phi} = \{\mathbf{\Psi}, \mathbf{\Theta}, R\}$ is the "fundamental" parameters' vector. At this stage, the standard calibration technique proceeds by simulating the model, and for this one needs to know $\mathbf{\Phi}$. Typically the problem is solved by importing the value of $\mathbf{\Phi}$ from other studies.

I propose to estimate $\mathbf{\Phi}$ using a distance metric specified in frequency domain. Let Y_{1t} and Y_{2t} denote actual and simulated univariate time series, respectively, and let

(3) $$f_1(e^{-i\omega}) = \sum_{s=-\infty}^{\infty} \gamma_1(s)\, e^{-i\omega s}$$

(4) $$f_2(e^{-i\omega}, \mathbf{\Phi}) = \sum_{s=-\infty}^{\infty} \gamma_2(s, \mathbf{\Phi})\, e^{-i\omega s}$$

be their corresponding spectral densities, where $0 \leq \omega \leq \pi$ is frequency measuring number of cycles per period, and $\gamma_1(s)$ and $\gamma_2(s, \mathbf{\Phi})$ are their auto-covariance functions given by

(5) $$\gamma_1(s) = E[(Y_{1,t+s} - \mu_1)(Y_{1,t} - \mu_1)]$$

(6) $$\gamma_2(s, \mathbf{\Phi}) = E[(Y_{2,t+s}(\mathbf{\Phi}) - \mu_2)(Y_{2,t}(\mathbf{\Phi}) - \mu_2)]$$

where μ_1 and μ_2 are the means of Y_{1t} and Y_{2t}, respectively. In (4), f_2 depends on $\mathbf{\Phi}$ because the simulated series and their auto-covariance (6) depend on the model's parameter values.

Now, consider the following metric, a distance function in frequency domain,

(7) $$d(\mathbf{\Phi}) = \int_{\omega_1}^{\omega_2} \lambda(\omega) \left| f_1\left(e^{-i\omega}\right) - f_2\left(e^{-i\omega}, \mathbf{\Phi}\right) \right|^{\xi} d\omega$$

where $\lambda(\omega)$ is a weight function, $\xi > 0$, $0 \leq \omega_1 \leq \pi$, $0 \leq \omega_2 \leq \pi$, and $\omega_1 < \omega_2$. I set $\xi = 1$ although $\xi = 2$ could also be considered.

The spectrum decomposes a series variance by frequency. That is,

$$\text{Var}(Y_t) = \int_{-\pi}^{\pi} f_y(e^{-i\omega})\, d\omega \,.$$

The metric $d(\mathbf{\Phi})$, therefore, measures the distance between the variances of the actual and simulated series *at each frequency*.

I propose to combine the standard calibration approach with a statistical estimation method, which is based on minimizing the frequency-weighted spectral density differential as given in (7). In other words, I suggest estimating the "deep" parameters' vector $\mathbf{\Phi}$, by minimizing $d(\mathbf{\Phi})$. The goal is to find model parameter values, which yield the best possible match between the spectral densities of the actual and model-generated time series.

3. The Weight Function

The function $\lambda(\omega)$ determines the weight attached to each frequency. The shape of the weight function depends on the goals of the researcher. For example, the frequency interval $[\omega_1, \omega_2]$ does not have to necessarily coincide with the entire frequency band $[0, \pi]$, which is what most practitioners of spectral analysis would typically consider. Under the current formulation, the interval $[\omega_1, \omega_2]$ can be a subset of the interval $[0, \pi]$. Moreover, $[\omega_1, \omega_2]$ can consist of more than one non-overlapping sub-intervals of $[0, \pi]$. This formulation introduces flexibility into the estimation methodology and makes the proposed approach more general. Below I describe some practically useful forms of the weight function.

Following Levy (1994) and Levy and Chen (1994), I divide the frequency interval into three bands: the long run, the business cycle, and the short-run. Business cycles are defined as 12–32 quarter (or 3–8 year) cycles. Therefore, $0.20 \le \omega \le 0.52$ corresponds to business cycle frequencies when quarterly data are used. The frequencies $\omega \le 0.20$ correspond to the long run, while the frequencies $\omega \ge 0.52$ correspond to the short-run.

3.1. Uniform Weighting

If we assign an equal weight to each frequency, then $\lambda(\omega) = 1/n$ for each ω, where n is the number of Fourier frequencies. Such a weight structure is behind the ordinary time-domain Gaussian regression analysis where all frequencies are implicitly assigned the same weight. Then (7) collapses into the standard metric used in the existing literature, namely the gap between the variances of the actual and simulated series, $d(\mathbf{\Phi}) = \text{Var}(Y_{1t}) - \text{Var}(Y_{2t})$. Thus, the standard distance metric is a particular case of the metric I propose.

3.2. Focusing on the Long Run

If we want to generate time series, which mimic the low frequency behavior of the actual data, then $\lambda(\omega)$ may be set so that we assign a high weight to low frequencies and a low weight (or a zero weight) to high frequencies. For example, we may set

$$(8) \qquad \lambda(\omega) = \begin{cases} 1 \text{ if } \ 0 \le \omega \le 0.20 \\ 0 \text{ if } \ 0.20 \le \omega \le \pi \end{cases}$$

3.3. Focusing on Business Cycles

If we want the generated series to mimic the cyclical behavior of the data, then we can set

$$(9) \qquad \lambda(\omega) = \begin{cases} 1 \text{ if } \ 0.20 \le \omega \le 0.52 \\ 0 \text{ if } \ \omega < 0.20 \text{or} \omega > 0.52 \end{cases}$$

3.4. Focusing on Seasonality

The estimation method I propose lends itself naturally to a study of seasonality. Define $\omega_s k$ as seasonal frequencies where $k = 1,2,\ldots,\lfloor N/2 \rfloor$, $\omega_s = 2\pi / N$, N is the number of observations taken in a one-year period, and $\lfloor N/2 \rfloor$ is the largest integer less than $N/2$. Thus, with monthly data, the exact seasonal frequencies are $2\pi k/12$, with $k = 1,2,\ldots,6$, which capture the deterministic component of seasonal fluctuations.

To allow deterministic as well as non-deterministic (stationary) seasonality as in Carpenter and Levy (1998), one needs to consider these seasonal components along with their neighboring frequency bands,

$$(10) \qquad \omega_s(\delta) = \{\omega \in [\omega_s k - \delta, \omega_s k + \delta], k = 1,2,\ldots,5, (\omega_s 6 - \delta, \pi)\}$$

where δ measures the width of the band.

3.5. Proportional Weighting

A potentially useful weight structure can be formed by requiring that each frequency be given a weight proportional to its contribution to the series' total variance. Thus, we can set

$$\lambda(\omega) = f_1(e^{-i\omega}) \Big/ \int_0^{\pi} f_1(e^{-i\omega}) d\omega \tag{11}$$

where $0 \le \omega \le \pi$, and the denominator equals the total variance of Y_{1t}. This could be a useful choice if the goal is to explore the general fit of the model at all frequencies.

4. Multivariate Extension

A multivariate extension of equations (3) and (4) are given by the matrix equations

$$\mathbf{F}_1(e^{-i\omega}) = \sum_{s=-\infty}^{\infty} \mathbf{\Gamma}_1(s) e^{-i\omega s} \tag{12}$$

$$\mathbf{F}_2(e^{-i\omega}, \mathbf{\Phi}) = \sum_{s=-\infty}^{\infty} \mathbf{\Gamma}_2(s, \mathbf{\Phi}) e^{-i\omega s} \tag{13}$$

where $\mathbf{F}_1(e^{-i\omega})$ and $\mathbf{F}_2(e^{-i\omega}, \mathbf{\Phi})$ are the spectral density matrices of the actual and simulated vector time series $\mathbf{Y}_1$ and $\mathbf{Y}_2$, and $\mathbf{\Gamma}_1(s)$ and $\mathbf{\Gamma}_2(s, \mathbf{\Phi})$ are their s^{th} lag auto-covariance matrices, respectively.

The diagonal elements of (12) and (13) are the spectral densities while the off-diagonal elements are the cross-spectral densities. Moreover, a multivariate extension of the distance metric (7) can be introduced by defining

$$\mathbf{D}(\mathbf{\Phi}) = \int_{\omega_1}^{\omega_2} \mathbf{\Lambda}(\omega) \otimes \left| \mathbf{F}_1\left(e^{-i\omega}\right) - \mathbf{F}_2\left(e^{-i\omega}, \mathbf{\Phi}\right) \right|^{\xi} d\omega \tag{14}$$

where $\mathbf{\Lambda}(\omega)$ is a weight matrix, and $\otimes$ is an element by element multiplication operator. Because $\mathbf{D}(\mathbf{\Phi})$ is a matrix, we can proceed by minimizing $\text{tr}[\mathbf{D}(\mathbf{\Phi})]$. The weight matrix can be redefined accordingly.

For example, a multivariate version of (11) will take the form

$$(15) \qquad \mathbf{\Lambda}(\omega) = \mathbf{F}_1\left(e^{-i\omega}\right) \varnothing \int_{-\pi}^{\pi} \mathbf{F}_1\left(e^{-i\omega}\right) d\omega$$

where $\varnothing$ is an element by element division operator. Other weight functions discussed above could be generalized similarly. Such a multivariate extension can also accommodate a co-integration analysis as discussed by Levy (2000, 2002).

5. Sampling Distribution of the Model Parameters and Testing the Model's Fit

To study the model parameters properties, we can examine their small sample distributions by repeatedly drawing the random variables driving the exogenous stochastic processes of the model with the same hyper-parameters. For model testing, we compare the data and the model spectra. Given Y_1 and Y_2, let $\hat{f}_1(e^{-i\omega})$ and $\hat{f}_2(e^{-i\omega}, \hat{\mathbf{\Phi}})$ be their estimated spectra. Then, the statistic

$$[\hat{f}_1(e^{-i\omega}) \big/ \hat{f}_2(e^{-i\omega}, \hat{\mathbf{\Phi}})] \sim \phi(\omega) F_{2,2},$$

where $\phi(\omega) = f_1(e^{-i\omega}) \big/ f_2(e^{-i\omega}, \mathbf{\Phi})$.

Under $\mathrm{H}_0 : \phi(\omega) = 1$, assuming that Y_1 and Y_2 follow the same distribution, a confidence interval may be formed around the ratio of the spectra using the fact that the ratio

$$[\hat{f}_1(e^{-i\omega}) \big/ \hat{f}_2(e^{-i\omega}, \hat{\mathbf{\Phi}})] \sim F_{2,2}.$$

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